



The Bright Side of Mathematics

Weierstrass M-test

→ comparison test for uniform convergence

Theorem: $\mathcal{D}$ set, $f_k: \mathcal{D} \rightarrow \mathbb{C}$ functions for each $k \in \mathbb{N}$.

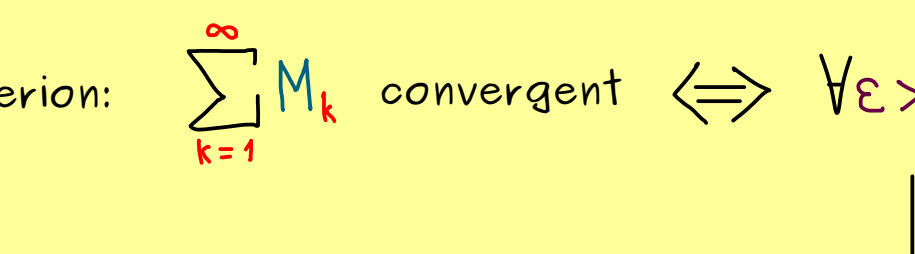
If there are constants $M_k \geq 0$ such that $\sum_{k=1}^{\infty} M_k < \infty$ and

$$|f_k(x)| \leq M_k \text{ for all } k \in \mathbb{N}, x \in \mathcal{D},$$

then: $\sum_{k=1}^{\infty} f_k$ is uniformly convergent $\left(\sum_{k=1}^{\infty} |f_k| \text{ is uniformly convergent} \right)$

More precisely: $S: \mathcal{D} \rightarrow \mathbb{C}$, $x \mapsto \sum_{k=1}^{\infty} f_k(x)$ is well-defined

$$\text{and } \left\| S - \sum_{k=1}^n f_k \right\|_{\infty} \xrightarrow{n \rightarrow \infty} 0$$



Proof: Cauchy criterion: $\sum_{k=1}^{\infty} M_k$ convergent $\Leftrightarrow \forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n \geq m \geq N:$

$$\left| \sum_{k=m}^n M_k \right| < \varepsilon$$

For a given $\varepsilon > 0$, we can choose $N \in \mathbb{N}$ as above.

For every $x \in \mathcal{D}$:

$$\left| \sum_{k=m}^n f_k(x) \right| \stackrel{\Delta\text{-inequality}}{\leq} \sum_{k=m}^n |f_k(x)| \leq \sum_{k=m}^n M_k < \varepsilon$$

$S: \mathcal{D} \rightarrow \mathbb{C}$, $x \mapsto \sum_{k=1}^{\infty} f_k(x)$ exists!

Hence: For a given $\varepsilon > 0$, we can choose $N \in \mathbb{N}$, $n \geq N$ as above.

$$\begin{aligned} \left\| S - \sum_{k=1}^n f_k \right\|_{\infty} &= \sup_{x \in \mathcal{D}} \left| S(x) - \sum_{k=1}^n f_k(x) \right| = \sup_{x \in \mathcal{D}} \left| \lim_{m \rightarrow \infty} \sum_{k=1}^m f_k(x) - \sum_{k=1}^n f_k(x) \right| \\ &= \sup_{x \in \mathcal{D}} \lim_{m \rightarrow \infty} \left| \sum_{k=1}^m f_k(x) - \sum_{k=1}^n f_k(x) \right| = \sup_{x \in \mathcal{D}} \lim_{m \rightarrow \infty} \underbrace{\left| \sum_{k=n+1}^m f_k(x) \right|}_{< \varepsilon} \\ &\leq \varepsilon \end{aligned}$$

$$\Rightarrow \left\| S - \sum_{k=1}^n f_k \right\|_{\infty} \xrightarrow{n \rightarrow \infty} 0 \quad \square$$

Example: $f_k: \mathbb{R} \rightarrow \mathbb{R}$, $f_k(x) = \frac{\cos(kx)}{k^2}$, $|f_k(x)| \leq \frac{1}{k^2}$ for all $x \in \mathbb{R}$

$$\sum_{k=1}^{\infty} \frac{1}{k^2} < \infty \stackrel{\text{Weierstrass M-Test}}{\Rightarrow} \sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2} \text{ converges uniformly}$$