Sylvester's Law of Inertia

Example of quadratic form: Take $\mathbb{C}^n$ with standard inner product:

$$x \longmapsto \langle x, x \rangle_{\text{standard}} = \sum_{i=1}^{n} \overline{x_i} \cdot x_i = \sum_{i=1}^{n} |x_i|^2$$

Definition: Let V be an $\mathbb{F}$-vector space $(\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\})$.

(1) A map $s: V \times V \longrightarrow \mathbb{F}$ is called sesquilinear form if:

$$s(u, v + \lambda w) = s(u, v) + \lambda \cdot s(u, w) \quad \text{(linear in the second argument)}$$

$$s(u + \lambda v, w) = s(u, w) + \overline{\lambda} \cdot s(v, w) \quad \text{(conjugate-linear in the first argument)}$$

(2) A sesquilinear form is called conjugate-symmetric (or Hermitian)

$$s(u, v) = \overline{s(v, u)}$$

(3) If s is a conjugate-symmetric sesquilinear form, then

$q: V \longrightarrow \mathbb{R}$, $\quad q(v) := s(v, v)$ is called the associated quadratic form.

Polarization identity: The associated quadratic form q determines s completely:

$$s(u, v) = \frac{1}{4}\Big(q(u+v) - q(u-v) \Big) \quad \text{for } \mathbb{F} = \mathbb{R}$$

$$s(u, v) = \frac{1}{4}\Big(q(u+v) - q(u-v) - i\, q(u+iv) + i\, q(u-iv) \Big) \quad \text{for } \mathbb{F} = \mathbb{C}$$

Matrix representation of a quadratic form:

$$\begin{tikzcd}
\boxed{V \text{ with basis } \mathcal{B} = (b_1, \dots, b_n)} \arrow[dr, bend left, "q"] \arrow[d, "\Phi_{\mathcal{B}}"', "\text{basis isomorphism}"] & \\
\boxed{\mathbb{F}^n \text{ with standard basis } (e_1, \dots, e_n)} \arrow[r, bend right, "q_{\mathcal{B}}"'] & \mathbb{R}
\end{tikzcd}$$

$$q_{\mathcal{B}}(x) = q\left(\Phi_{\mathcal{B}}^{-1}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}\right) = q\left(\sum_{j=1}^{n} x_j \cdot b_j\right) = s\left(\sum_{i=1}^{n} x_i \cdot b_i \,,\, \sum_{j=1}^{n} x_j \cdot b_j\right)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \overline{x_i} \cdot x_j \underbrace{s(b_i, b_j)}_{=: a_{ij}} = \langle x, Ax \rangle_{\text{standard}}$$

self-adjoint matrix $A = (a_{ij})$

Diagonalization of quadratic forms:

Let V be an $\mathbb{F}$-vector space, s be a conjugate-symmetric sesquilinear form, and $q: V \longrightarrow \mathbb{R}$ be the associated quadratic form.

Then there is a basis $\mathcal{B} = (b_1, \dots, b_n)$ such that $q_{\mathcal{B}}(x) = \langle x, Ax \rangle_{\text{standard}}$ is given by a diagonal matrix:

$$A = \begin{pmatrix} \underbrace{1 \; \ddots \; 1}_{r_+ \text{ times}} & & \\ & \underbrace{-1 \; \ddots \; -1}_{r_- \text{ times}} & \\ & & \underbrace{0 \; \ddots \; 0}_{r_0 \text{ times}} \end{pmatrix}$$

Proof: Case $q = 0$ ✓

Induction base case: $n = 1$: Choose $v \in V \setminus \{0\}$ with $q(v) \neq 0$.

Define $w := \dfrac{v}{\sqrt{|q(v)|}} \rightsquigarrow \mathcal{B} = (w)$

We get: $q(w) = s(w, w) = \dfrac{1}{|q(v)|} \cdot q(v) = \begin{cases} +1 \\ -1 \end{cases}$

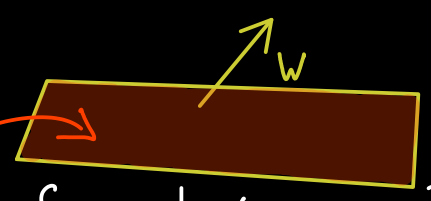
$$\langle x, Ax \rangle_{\text{standard}} = q\left(\Phi_{\mathcal{B}}^{-1}(x)\right) = q(x \cdot w) = \begin{cases} +|x|^2 \\ -|x|^2 \end{cases}$$

$$\| $$

$$a \cdot |x|^2 \implies a = 1 \text{ or } a = -1 \quad ✓$$

$\underline{n-1 \longrightarrow n}$: Choose $v \in V \setminus \{0\}$ with $q(v) \neq 0$. $\quad w := \dfrac{v}{\sqrt{|q(v)|}}$

So $q(w) = \begin{cases} +1 \\ -1 \end{cases}$



$U := \{ u \in U \mid s(w,u) = 0 \}$

kernel of linear map: $s(w,\cdot) : V \longrightarrow \mathbb{F}$

with dimension $n-1$ by rank-nullity theorem

$\Longrightarrow$ there is a basis of U: $\mathcal{B} = (b_1, \dots, b_{n-1})$

such that

$$(q|_U)_{\mathcal{B}}(x) = \left\langle x, \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & -1 & & & \\ & & & & \ddots & & \\ & & & & & -1 & \\ & & & & & & 0 \\ & & & & & & & \ddots \\ & & & & & & & & 0 \end{pmatrix} x \right\rangle_{\text{standard}}$$

Take w to extend basis to $\tilde{\mathcal{B}} = (w, b_1, \dots, b_{n-1})$ (case $q(w) = +1$)

$$q_{\tilde{\mathcal{B}}}(y) = q\Big(y_0 w + \sum_{j=1}^{n-1} y_j b_j\Big) = s(y_0 w, y_0 w) + s\Big(\sum_{i=1}^{n-1} y_i b_i, \sum_{j=1}^{n-1} y_j b_j\Big)$$

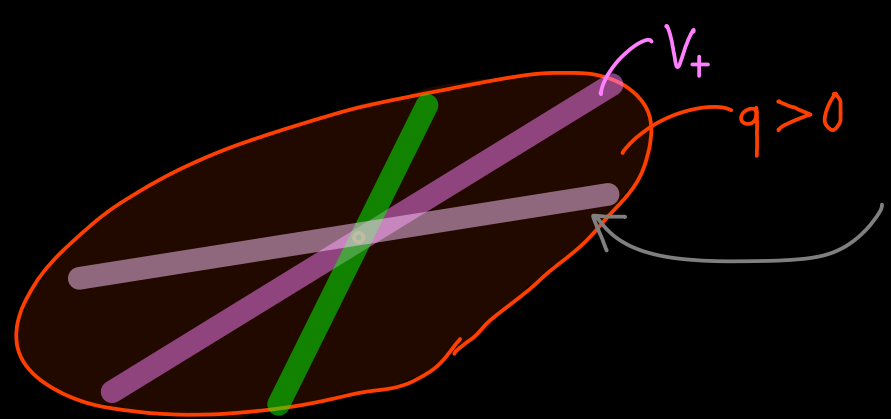
$$y = \begin{pmatrix} y_0 \\ \vdots \\ y_{n-1} \end{pmatrix}$$

$$= |y_0|^2 \underbrace{q(w)}_{=1} + \left\langle \begin{pmatrix} y_1 \\ \vdots \\ y_{n-1} \end{pmatrix}, \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & -1 & & & \\ & & & & \ddots & & \\ & & & & & -1 & \\ & & & & & & 0 \\ & & & & & & & \ddots \\ & & & & & & & & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_{n-1} \end{pmatrix} \right\rangle$$

$$= \left\langle \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{pmatrix}, \begin{pmatrix} +1 & & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ & & & & -1 & & \\ & & & & & \ddots & \\ & & & & & & -1 \\ & & & & & & & 0 \\ & & & & & & & & \ddots \\ & & & & & & & & & 0 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{pmatrix} \right\rangle_{\text{standard}} \qquad \square$$

Note: The basis $(\underbrace{b_1, \dots, b_{r_+}}_{V_+}, \underbrace{b_{r_++1}, \dots, b_{r_++r_-}}_{V_-}, \underbrace{b_{r_++r_-+1}, \dots, b_n}_{V_0})$

gives subspaces: $V_+ \to q > 0$ (expect zero vector), $V_- \to q < 0$ (expect zero vector), $V_0 \to q = 0$



What is the largest dimension that fits in?

Theorem: (Sylvester's law of inertia)

The numbers r_+, r_-, r_0 in the diagonalization are invariant:

they don't change if we choose a different basis that diagonalizes q.

Moreover: $r_+ = \max\left\{ \dim(U) \;\middle|\; U \subseteq \{v \in V \mid q(v) > 0\} \cup \{0\},\ U \text{ subspace of } V \right\}$

$r_- = \max\left\{ \dim(U) \;\middle|\; U \subseteq \{v \in V \mid q(v) < 0\} \cup \{0\},\ U \text{ subspace of } V \right\}$

Proof: Take basis $(\underbrace{b_1, \dots, b_{r_+}}_{\text{span } V_+}, \underbrace{b_{r_++1}, \dots, b_{r_++r_-}, b_{r_++r_-+1}, \dots, b_n}_{\text{span } V_- \oplus V_0})$

Now assume that there is subspace $W_+ \subseteq \{v \in V \mid q(v) > 0\} \cup \{0\}$

with $\dim(W_+) > \dim(V_+)$.

Then there is $w \in W_+ \setminus \{0\}$ with $w \in V_- \oplus V_0$: $w = \sum_{j=r_++1}^{n} \lambda_j b_j$

$$\Rightarrow q(w) = q_{\mathcal{B}}(\Phi_{\mathcal{B}}(w))$$

$$= \left\langle \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \lambda_{r_++1} \\ \vdots \\ \lambda_n \end{pmatrix}, \begin{pmatrix} 1 & & & & & & & & \\ & 1 & & & & & & & \\ & & \ddots & & & & & & \\ & & & 1 & & & & & \\ & & & & -1 & & & & \\ & & & & & \ddots & & & \\ & & & & & & -1 & & \\ & & & & & & & 0 & \\ & & & & & & & & \ddots \\ & & & & & & & & & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \lambda_{r_++1} \\ \vdots \\ \lambda_n \end{pmatrix} \right\rangle \leq 0 \ \lightning$$

$\square$

Definition: We call $r_+ + r_-$ <u>rank</u> and $r_+ - r_-$ <u>signature</u> of the quadratic form.