

Introduction

Definition 1. Real numbers

The real numbers are a non-empty set $\mathbb{R}$ together with the operations $+: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ and $\cdot: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ and an ordering relation $\leq: \mathbb{R} \times \mathbb{R} \rightarrow \{\text{True}, \text{False}\}$ that fulfil the following rules

(A) Addition

(A1) associative: $x + (y + z) = (x + y) + z$

(A2) neutral element: There is a (unique) element 0 with $x + 0 = x$ for all x .

(A3) inverse element: For all x there is a (unique) y with $x + y = 0$. We write for this element simply $-x$.

(A4) commutative: $x + y = y + x$

(M) Multiplication

(M1) associative: $x \cdot (y \cdot z) = (x \cdot y) \cdot z$

(M2) neutral element: There is a (unique) element $1 \neq 0$ with $x \cdot 1 = x$ for all x .

(M3) inverse element: For all $x \neq 0$ there is a (unique) y with $x \cdot y = 1$. We write for this element simply x^{-1} .

(M4) commutative: $x \cdot y = y \cdot x$

(D) Distributivity: $x \cdot (y + z) = x \cdot y + x \cdot z$.

(O) Ordering

(O1) $x \leq x$ is true for all x .

(O2) If $x \leq y$ and $y \leq x$, then $x = y$.

(O3) transitive: $x \leq y$ and $y \leq z$ imply $x \leq z$.

(O4) For all $x, y \in X$, we have either $x \leq y$ or $y \leq x$.

(O5) $x \leq y$ implies $x + z \leq y + z$ for all z .

(O6) $x \leq y$ implies $x \cdot z \leq y \cdot z$ for all $z \geq 0$.

(O7) $x > 0$ and $\varepsilon > 0$ implies $x < \varepsilon + \dots + \varepsilon$ for sufficiently many summands.

(C) Let $X, Y \subset \mathbb{R}$ be two non-empty subsets with the property $x \leq y$ for all $x \in X$ and $y \in Y$. Then there is a $c \in \mathbb{R}$ with $x \leq c \leq y$ for all $x \in X$ and $y \in Y$.

Remark 2.

In the video, we described the completeness axiom with the help of sequences. Then it sounds like:

Completeness: Every sequence $(a_n)_{n \in \mathbb{N}}$ with the property [For all $\varepsilon > 0$ there is an $N \in \mathbb{N}$ with $|a_n - a_m| < \varepsilon$ for all $n, m > N$] has a limit.

Later, we will show the equivalence of both descriptions.

Definition 3. Absolute value for real numbers

The **absolute value** of a number $x \in \mathbb{R}$ is defined by

$$|x| := \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

Exercise 4.

Use the axioms to show:

$$(1) : 0 \cdot x = 0$$

$$(2) : -x = (-1) \cdot x$$

$$(3) : (-1) \cdot (-1) = 1$$

$$(4) : 1 > 0$$

$$(1): 0 \cdot x \stackrel{(A2)}{=} (0 + 0)x \stackrel{(D)}{=} 0x + 0x \stackrel{(A2)}{\implies} 0x = 0$$

$$(2): -x \stackrel{(*)}{=} 0x + (-x) \stackrel{(A3)}{=} (1 + (-1))x + (-x) \stackrel{(D)}{=} x + (-1)x + (-x) \stackrel{(A2-4)}{=} (-1)x$$

$$(3): x(-1)(-1) \stackrel{(**)}{=} -(-x) \stackrel{(A3)}{=} x \stackrel{(M2)}{\implies} (-1) \cdot (-1) = 1$$

(4): Try for yourself.

Exercise 5. Learn to sketch

Sketch the following subsets of $\mathbb{R}$:

$$(a) A := \{x \in \mathbb{R} \mid x < x^2\}$$

$$(b) B := \{x \in \mathbb{R} \mid x > 0 \wedge \frac{10}{x} - 3 \leq \frac{4}{x} + 1\}$$

Exercise 6. Sketch in two dimensions

Sketch the following subsets of $\mathbb{R}^2$:

$$(a) M_1 := \{(x, y) \in \mathbb{R}^2 \mid (x - 3)^2 + (y + 4)^2 \leq 9\}$$

$$(b) M_2 := \{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 \geq 1\}$$

Exercise 7. Use the axioms!

Prove for $x, y, z \in \mathbb{R}$ the following statements by using the axioms of the real numbers:

$$(a) x \leq y \wedge z \leq 0 \Rightarrow xz \geq yz$$

$$(b) ||x| - |y|| \leq |x - y|$$

$$(c) |xy| \leq \frac{1}{2}(x^2 + y^2)$$

$$(d) x, y \geq 0 \Rightarrow xy \leq \frac{1}{4}(x + y)^2$$