



QR-decomposition

Let A be square matrix + invertible

$$A = Q \cdot R \quad \leftarrow \begin{pmatrix} \text{upper triangular} \\ 0 \end{pmatrix} \quad \left[\begin{array}{l} \text{related to} \\ \text{Gram-Schmidt} \\ \text{process} \end{array} \right]$$

$Q^T = Q^{-1} \leftarrow \begin{array}{l} \text{orthogonal /} \\ \text{unitary matrix} \end{array} \rightsquigarrow Q^* = Q^{-1}$
(columns of Q form ONB)

$$A = \begin{pmatrix} | & | & | \\ a_1 & a_2 & a_3 \\ | & | & | \end{pmatrix} \in \mathbb{R}^{3 \times 3} \rightsquigarrow Q = \begin{pmatrix} | & | & | \\ q_1 & q_2 & q_3 \\ | & | & | \end{pmatrix}$$

$$\textcircled{1} \quad q_1 := \frac{a_1}{\|a_1\|} \quad (\text{hence: } \|q_1\| = 1) \quad R = \begin{pmatrix} \square & \square & \square \\ 0 & \square & \square \\ 0 & 0 & \square \end{pmatrix}$$

$$\textcircled{2} \quad a_2^\perp = a_2 - \langle a_2, q_1 \rangle q_1$$

$$q_2 := \frac{a_2^\perp}{\|a_2^\perp\|}$$

$$\textcircled{3} \quad a_3^\perp = a_3 - \langle a_3, q_1 \rangle q_1 - \langle a_3, q_2 \rangle q_2$$

$$q_3 := \frac{a_3^\perp}{\|a_3^\perp\|}$$

Example: $A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}, \quad R = \begin{pmatrix} \sqrt{2} & \sqrt{2} \\ 0 & 2\sqrt{2} \end{pmatrix}$

$$\textcircled{1} \quad q_1 = \frac{a_1}{\|a_1\|} = \frac{1}{\sqrt{1^2+1^2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\textcircled{2} \quad a_2^\perp = a_2 - \langle a_2, q_1 \rangle q_1 = \begin{pmatrix} 3 \\ -1 \end{pmatrix} - \left\langle \begin{pmatrix} 3 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} (3-1) = \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$q_2 = \frac{1}{\underbrace{\sqrt{2^2+(-2)^2}}_{=2\sqrt{2}}} \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} - \sqrt{2} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

Another example:

$$A = \begin{pmatrix} 2 & -2 & -12 \\ 4 & 2 & -18 \\ -4 & -8 & 30 \end{pmatrix}$$