

Dealing with Sets and Maps

Exercise 1. Sets, Products, Functions

1. Complete the following sentences:

- a) The set $A := \{x \in \mathbb{R} : (x+1) \in (-\infty, 0]\}$ contains all x of $\mathbb{R}$ such that ...
- b) The set $B := \{1 - \exp(-x) : |x| > 4\}$ contains all ...
- c) The set $C := \{(x, y) \in [0, \infty) \times \mathbb{R} : x \leq y\}$ contains all pairs of numbers (x, y) where $x \in \dots$ and $y \in \dots$ such that
- d) x is an element of the set $\bigcap_{n=1}^{\infty} [0, 1/n]$ if $\dots \leq \dots \leq \dots$ for all In particular $\dots = 0$.
- e) y is an element of the set $\bigcup_{n \in \mathbb{N}} [n, n+1)$ if there exists $\dots \in \dots$ such that $\dots \leq \dots < \dots$.
- f) Let (a_n) and (b_n) be sequences in $\mathbb{R}$ and $\Omega \subseteq \mathbb{R}$. We want to prove the following equality of sets:

$$\bigcap_{n \in \mathbb{N}} \left((a_n, b_n) \cap \Omega \right) = \left(\bigcap_{n \in \mathbb{N}} (a_n, b_n) \right) \cap \Omega.$$

We first show that $x \in \bigcap_{n \in \mathbb{N}} \left((a_n, b_n) \cap \Omega \right) \implies x \in \left(\bigcap_{n \in \mathbb{N}} (a_n, b_n) \right) \cap \Omega$:

- i. Let $x \in \bigcap_{n \in \mathbb{N}} \left((a_n, b_n) \cap \Omega \right)$.
- ii. Then we have that $x \in \dots$ for all $n \in \mathbb{N}$.
- iii. This shows that $x \in \dots$ and $x \in \Omega$ for all $n \in \mathbb{N}$.
- iv. This implies that $x \in \dots$ and $x \in \Omega$ which allows us to conclude that the inclusion $x \in \left(\bigcap_{n \in \mathbb{N}} (a_n, b_n) \right) \cap \Omega$ holds.

Now, use a similar proof to show the converse statement.

2. Define a suitable set that matches the following descriptions:

- a) A is the set of all open real intervals with rational endpoints.
- b) If $f: \mathbb{R} \rightarrow \mathbb{R}$ is a function, then B is the set of all $x \in \mathbb{R}$ such that $|f(x)|$ is smaller than 5.
- c) C is the set of all real numbers smaller or equal 4 and strictly greater than 3.
- d) D is the set of all vectors of length 3 with positive integer entries, such that the sum of the entries equals 9.

For each of the sets A , B , C and D give an example of an element of the respective set.

Exercise 2. Functions, Images and Preimages

(a) Let $f: \mathbb{R} \rightarrow \mathbb{R}$ and $a \in \mathbb{R}$. Which of the following sets coincide?

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|---|--|
| a) $f^{-1}([0, \infty))$ | f) $\{f(x): 0 \leq x < \infty\}$ |
| b) $\{f(x): x \in [0, \infty)\}$ | g) $f(\{x \in \mathbb{R}: x \geq 0\})$ |
| c) $\{f(x): (x+a) \in [a, \infty)\}$ | h) $f([0, \infty))$ |
| d) $\{x \in \mathbb{R}: f(x) \in [0, \infty)\}$ | i) $\{y \in f(\mathbb{R}): y \geq 0\}$ |
| e) $\{f(x): x \geq 0 \text{ and } x < \infty\}$ | |

(b) Let $f: \mathbb{R} \rightarrow \mathbb{R}$ and $a > 0$. Which of the following sets coincide?

- | | |
|---|--|
| a) $f^{-1}([-a, a])$ | f) $\{f(x): x \leq a\}$ |
| b) $\{x \in \mathbb{R}: f(x) \leq a\}$ | g) $\{f(x): x \in [0, a]\}$ |
| c) $\{x \in \mathbb{R}: \max\{f(x), -f(x)\} \leq a\}$ | h) $f([a, \infty)) \cap f((-\infty, a])$ |
| d) $f(\mathbb{R})$ | i) $\{f(x): x \geq -a \text{ and } x \leq a\}$ |
| e) $f^{-1}([-a, \infty)) \cap f^{-1}((-\infty, a])$ | j) $\{y \in \mathbb{R}: \exists x \in \mathbb{R} \text{ with } f(x) = y\}$ |

(c) Let $f: X \rightarrow Y$ and $A \subseteq X$ and $B \subseteq Y$. Which of the following sets coincide?

- | | |
|--|--|
| a) $\{(x, y) \in X \times Y: \text{there exists } x \in A \text{ such that } f(x) = y\}$ | e) $f^{-1}(B) \times Y$ |
| b) $\{(x, y): x \in X, y \in Y\}$ | f) $\{(x, y) \in X \times Y: f(x) \in Y\}$ |
| c) $\{(x, y) \in X \times Y: y \in f(A)\}$ | g) $X \times Y$ |
| d) $\{(x, y) \in X \times Y: f(x) \in B \text{ and } f(x) = y\}$ | h) $\{(x, y) \in A \times Y: y \in B\}$ |
| | i) $A \times B$ |
| | j) $X \times f(A)$ |