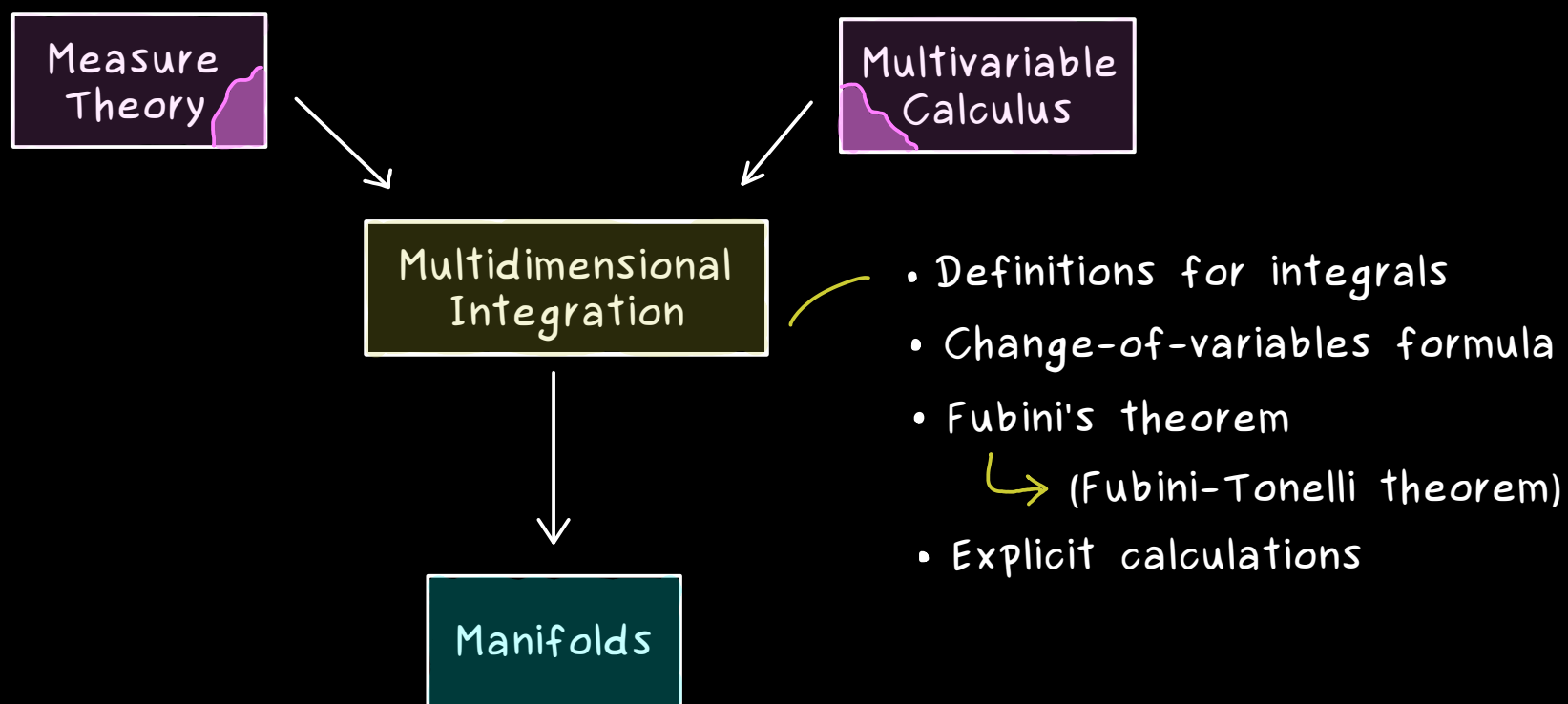
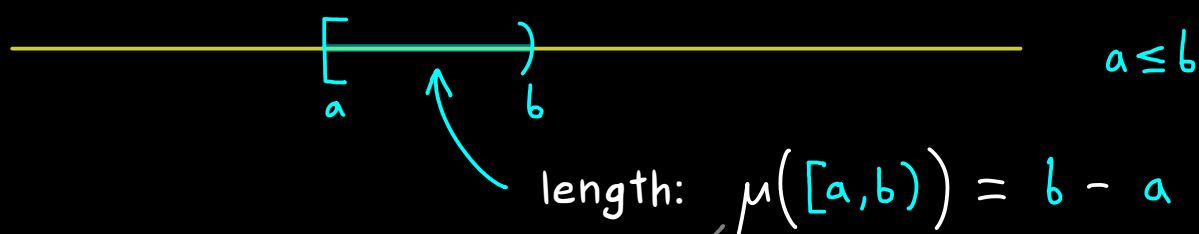


## Multidimensional Integration - Part 1



Lebesgue measure on  $\mathbb{R}$ :



Carathéodory's Extension Theorem

pre-measure

$$\psi: \mathcal{P}(\mathbb{R}) \longrightarrow [0, \infty]$$

outer measure

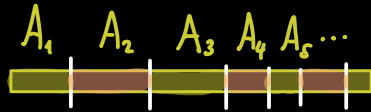
$\mathcal{L}(\mathbb{R}) = \mathcal{A}_\psi$   $\sigma$ -algebra of Lebesgue-measurable sets

$$\lambda: \mathcal{L}(\mathbb{R}) \longrightarrow [0, \infty] \quad \text{Lebesgue measure on } \mathbb{R}$$

measure

Properties:

- $\lambda(\emptyset) = 0$

- $\lambda\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \lambda(A_j)$  for  $A_j \in \mathcal{L}(\mathbb{R})$    
 $A_j \cap A_i = \emptyset$  for  $i \neq j$

- $\mathcal{L}(\mathbb{R})$  is larger than the Borel  $\sigma$ -algebra.

- If  $A \in \mathcal{L}(\mathbb{R})$  with  $\lambda(A) = 0$ , ( $A$  is called null set)  
then each  $B \subseteq A$  satisfies  $B \in \mathcal{L}(\mathbb{R})$ .

- $\lambda([a, b]) = b - a$ ,  $b \geq a$

- $\lambda(x + A) = \lambda(A)$  for all  $x \in \mathbb{R}$ ,  $A \in \mathcal{L}(\mathbb{R})$

(translation-invariant)

Definition (Lebesgue integral):

$$\int_A f d\lambda = \int_A f(x) d\lambda(x) = \int_A f(x) dx$$

defined by approximation with simple functions:

