



Jordan normal form - part 1

A square matrix diagonalisable $:\Leftrightarrow$

There is an invertible X with $X^{-1}AX = D$ ↖ diagonal matrix

$$A = X \underbrace{D}_{\text{Jordan normal form}} X^{-1} \quad (\text{matrix decomposition})$$

$A \in \mathbb{C}^{n \times n} \Rightarrow$ There exists a Jordan normal form J for A : $A = X J X^{-1}$

Example: $A \in \mathbb{C}^{9 \times 9}$ and with eigenvalues: $\{ \underbrace{2, 2, 2}_{\substack{\text{algebraic} \\ \text{multiplicity} \\ 3}}, \underbrace{3, 3, 3, 3}_{\substack{\text{algebraic} \\ \text{multiplicity} \\ 4}}, \underbrace{4, 4}_{\substack{\text{algebraic} \\ \text{multiplicity} \\ 2}} \}$

$$J = \begin{pmatrix} \boxed{\begin{matrix} 2 & & \\ & 2^{(*)} & \\ & & 2 \end{matrix}} & & \\ & \boxed{\begin{matrix} 3 & & \\ & 3 & \\ & & 3^{(*)} \\ & & & 3 \end{matrix}} & \\ & & \boxed{\begin{matrix} 4^{(*)} & \\ & 4 \end{matrix}} \end{pmatrix} \quad \text{three Jordan blocks}$$

$$\boxed{\begin{matrix} 2 & & \\ & 2^{(*)} & \\ & & 2 \end{matrix}}$$

:

(1)

$$\boxed{\begin{matrix} 2 & & \\ & 2 & \\ & & 2 \end{matrix}}$$

three Jordan boxes, geometric multiplicity: 3

(2)

$$\boxed{\begin{matrix} 2 & 1 & \\ & 2 & \\ & & 2 \end{matrix}}$$

two Jordan boxes, geometric multiplicity: 2

(3)

$$\boxed{\begin{matrix} 2 & 1 & \\ & 2 & 1 \\ & & 2 \end{matrix}}$$

one Jordan boxes, geometric multiplicity: 1

$$\begin{pmatrix} 3 & & & \\ & 3 & & \\ & & 3 & \\ & & & 3 \end{pmatrix} (*)$$

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(1) $\begin{pmatrix} 3 & & & \\ & 3 & & \\ & & 3 & \\ & & & 3 \end{pmatrix}$

four Jordan boxes , geometric multiplicity 4

(2) $\begin{pmatrix} 3 & 1 & & \\ & 3 & & \\ & & 3 & \\ & & & 3 \end{pmatrix}$

three Jordan boxes , geometric multiplicity 3

(3) $\begin{pmatrix} 3 & 1 & & \\ & 3 & & \\ & & 3 & 1 \\ & & & 3 \end{pmatrix}$

(4) $\begin{pmatrix} 3 & 1 & & \\ & 3 & 1 & \\ & & 3 & \\ & & & 3 \end{pmatrix}$

two Jordan boxes
geometric multiplicity 2

(5) $\begin{pmatrix} 3 & 1 & & \\ & 3 & 1 & \\ & & 3 & 1 \\ & & & 3 \end{pmatrix}$

one Jordan boxes , geometric multiplicity 1

$$XJX^{-1} = A$$

Recipe : (1) Calculate eigenvalues of A : $\lambda_1, \lambda_2, \dots, \lambda_k$

For $i=1..k$: (2) Determine algebraic multiplicity of λ_i and
geometric multiplicity of $\lambda_i := \dim(\text{Ker}(A - \lambda_i I))$

(3) If needed: $\dim(\text{Ker}(A - \lambda_i I)^2)$, $\dim(\text{Ker}(A - \lambda_i I)^3) \dots$