

The Bright Side of Mathematics



Jordan normal form - part 1

A square matrix diagonalisable : $\Leftrightarrow$

There is an invertible X with $X^{-1}AX = D$ ↖ diagonal matrix

$$A = \underbrace{X D X^{-1}}_{\text{Jordan normal form}} \quad (\text{matrix decomposition})$$

$A \in \mathbb{C}^{n \times n} \Rightarrow$ There exists a Jordan normal form J for A : $A = X J X^{-1}$

Example: $A \in \mathbb{C}^{9 \times 9}$ and with eigenvalues: $\{ \underbrace{2, 2, 2}_{\text{algebraic multiplicity } 3}, \underbrace{3, 3, 3, 3}_{\text{algebraic multiplicity } 4}, \underbrace{4, 4}_{\text{algebraic multiplicity } 2} \}$

$$J = \begin{pmatrix} \boxed{\begin{matrix} 2 & & \\ & 2^{(*)} & \\ & & 2 \end{matrix}} & & \\ & \boxed{\begin{matrix} 3 & & & \\ & 3 & & \\ & & 3^{(*)} & \\ & & & 3 \end{matrix}} & \\ & & \boxed{\begin{matrix} 4^{(*)} & \\ & 4 \end{matrix}} \end{pmatrix} \quad \text{three Jordan blocks}$$

$$\boxed{\begin{matrix} 2 & & \\ & 2^{(*)} & \\ & & 2 \end{matrix}} : \quad (1) \quad \boxed{\begin{matrix} 2 & & \\ & 2 & \\ & & 2 \end{matrix}} \quad \text{three Jordan boxes, geometric multiplicity: 3}$$

$$(2) \quad \boxed{\begin{matrix} 2 & 1 & \\ & 2 & \\ & & 2 \end{matrix}} \quad \text{two Jordan boxes, geometric multiplicity: 2}$$

$$(3) \quad \boxed{\begin{matrix} 2 & 1 & \\ & 2 & 1 \\ & & 2 \end{matrix}} \quad \text{one Jordan boxes, geometric multiplicity: 1}$$

$$\boxed{\begin{matrix} 3 & & & \\ & 3 & & \\ & & 3^{(*)} & \\ & & & 3 \end{matrix}} : \quad (1) \quad \boxed{\begin{matrix} 3 & & & \\ & 3 & & \\ & & 3 & \\ & & & 3 \end{matrix}} \quad \text{four Jordan boxes, geometric multiplicity 4}$$

$$(2) \quad \boxed{\begin{matrix} 3 & 1 & \\ & 3 & \\ & & 3 & \\ & & & 3 \end{matrix}} \quad \text{three Jordan boxes, geometric multiplicity 3}$$

$$(3) \quad \boxed{\begin{matrix} 3 & 1 & \\ & 3 & \\ & & 3 & 1 \\ & & & 3 \end{matrix}} \quad (4) \quad \boxed{\begin{matrix} 3 & 1 & & \\ & 3 & 1 & \\ & & 3 & \\ & & & 3 \end{matrix}} \quad \text{two Jordan boxes, geometric multiplicity 2}$$

$$(5) \quad \boxed{\begin{matrix} 3 & 1 & & & \\ & 3 & 1 & & \\ & & 3 & 1 & \\ & & & 3 & 1 \\ & & & & 3 \end{matrix}} \quad \text{one Jordan boxes, geometric multiplicity 1}$$

$$\boxed{X J X^{-1} = A}$$

Recipe: (1) Calculate eigenvalues of A : $\lambda_1, \lambda_2, \dots, \lambda_k$

For $i=1..k$: (2) Determine algebraic multiplicity of λ_i and geometric multiplicity of $\lambda_i := \dim(\text{Ker}(A - \lambda_i I))$

(3) If needed: $\dim(\text{Ker}(A - \lambda_i I)^2), \dim(\text{Ker}(A - \lambda_i I)^3) \dots$