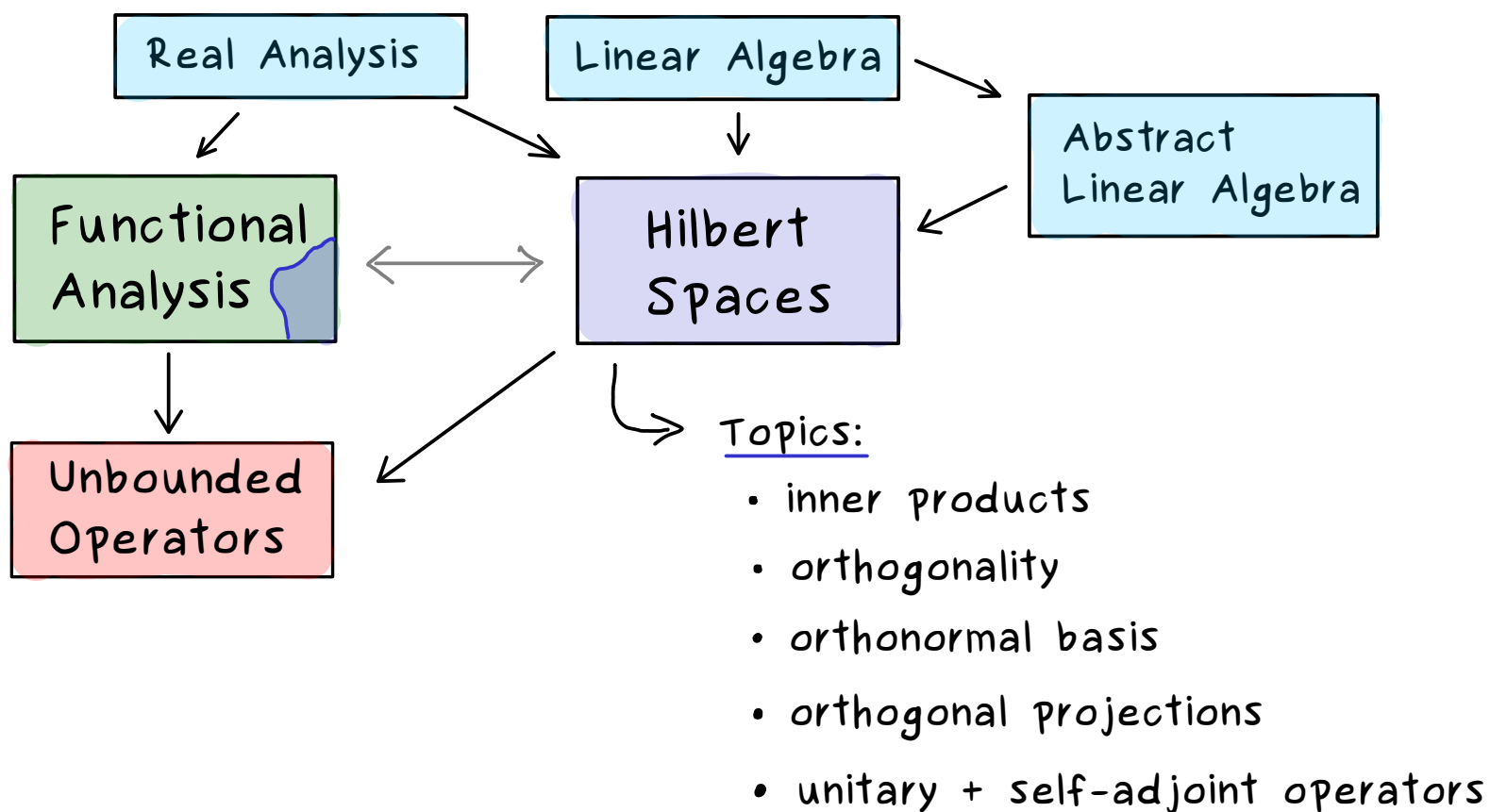


Hilbert Spaces - Part 1



Definition: $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. An $\mathbb{F}$ -vector space X with inner product $\langle \cdot, \cdot \rangle : X \times X \rightarrow \mathbb{F}$, which means

$$(1) \quad \langle x, x \rangle \geq 0 \quad \text{for all } x \in X \quad \text{(positive definite)}$$

$$\text{and } \langle x, x \rangle = 0 \implies x = 0 \quad \text{(zero vector)}$$

$$(2) \quad \langle y, x + \tilde{x} \rangle = \langle y, x \rangle + \langle y, \tilde{x} \rangle \quad \text{for all } x, \tilde{x}, y \in X$$

$$\langle y, \lambda \cdot x \rangle = \lambda \cdot \langle y, x \rangle \quad \text{for all } \lambda \in \mathbb{F}, x, \tilde{x}, y \in X$$

(linear in the second argument)

$$(3) \quad \langle x, y \rangle = \overline{\langle y, x \rangle} \quad \text{for all } x, y \in X \quad \text{(conjugate symmetric)}$$

is called an inner-product space. (pre-Hilbert space)

Cauchy-Schwarz inequality: For an inner product space $(X, \langle \cdot, \cdot \rangle)$, we have:

$$|\langle y, x \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle \quad \text{for all } x, y \in X$$

Proof: For $y \neq 0$:

$$\begin{aligned} 0 &\leq \left\langle x - \frac{\langle y, x \rangle}{\langle y, y \rangle} y, x - \frac{\langle y, x \rangle}{\langle y, y \rangle} y \right\rangle \\ &= \langle x, x \rangle - \cancel{\frac{\overline{\langle y, x \rangle}}{\langle y, y \rangle} \langle y, x \rangle} - \frac{\langle y, x \rangle}{\langle y, y \rangle} \langle x, y \rangle \\ &\quad + \cancel{\frac{\overline{\langle y, x \rangle}}{\langle y, y \rangle} \cdot \frac{\langle y, x \rangle}{\langle y, y \rangle} \langle y, y \rangle} \\ &= \langle x, x \rangle - \frac{|\langle y, x \rangle|^2}{\langle y, y \rangle} \end{aligned}$$

□

Result: $\|x\| := \sqrt{\langle x, x \rangle}$ defines a norm on X

Definition: An inner product space $(X, \langle \cdot, \cdot \rangle)$ is called a Hilbert space if $(X, \|\cdot\|)$ is complete.