

Cauchy-Schwarz for Sesquilinear Forms

Definition: Let X be a complex vector space. A map $s: X \times X \longrightarrow \mathbb{C}$ is called a sesquilinear form if

$$(1) \quad s(y, x_1 + \alpha x_2) = s(y, x_1) + \alpha s(y, x_2) \quad \text{for all } y, x_1, x_2 \in X \\ \text{for all } \alpha \in \mathbb{C}$$

$$(2) \quad s(y_1 + \beta y_2, x) = s(y_1, x) + \bar{\beta} s(y_2, x) \quad \text{for all } x, y_1, y_2 \in X \\ \text{for all } \beta \in \mathbb{C}$$

Special sesquilinear forms: If $s(y, x) = \overline{s(x, y)}$, then s is called conjugate-symmetric.
 $\implies$ Hermitian

If $s(x, x) \geq 0$, then s is called positive semidefinite.

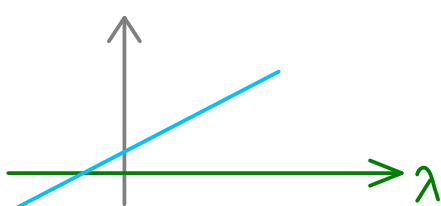
Cauchy-Schwarz inequality: Let s be a sesquilinear form that is conjugate-symmetric and positive semidefinite. Then for all $x, y \in X$:

$$|s(y, x)|^2 \leq s(x, x) \cdot s(y, y)$$

Proof: Take $x, y \in X$ and $\lambda \in \mathbb{R}$:

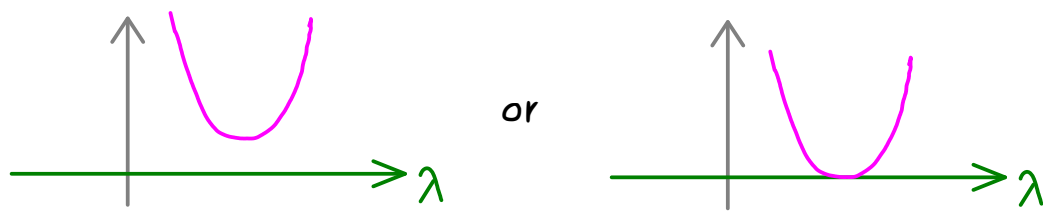
$$0 \leq s(y - \lambda x, y - \lambda x) = s(y, y) - \lambda s(x, y) - \lambda s(y, x) + \lambda^2 s(x, x) \\ = s(y, y) + \lambda(-2 \operatorname{Re} s(y, x)) + \lambda^2 s(x, x)$$

If $s(x, x) = 0$, then $\operatorname{Re} s(y, x) = 0$.



$$\text{If } s(x, x) \neq 0, \text{ then } 0 \leq \underbrace{\frac{s(y, y)}{s(x, x)}}_q + \lambda \underbrace{\left(\frac{-2 \operatorname{Re} s(y, x)}{s(x, x)} \right)}_p + \lambda^2$$

quadratic polynomial has zeros: $\lambda_{1,2} = -\frac{p}{2} \pm \sqrt{\underbrace{\left(\frac{p}{2}\right)^2 - q}_{\leq 0}}$



We get: $\left(\frac{\text{Re } s(y, x)}{s(x, x)} \right)^2 \leq \frac{s(y, y)}{s(x, x)}$

In both cases: $\operatorname{Re} s(y, x)^2 \leq s(x, x) s(y, y) \quad (*)$

Go to the absolute value: Take $e^{i\psi} = c$ such that $c \cdot s(y, x) = |s(y, x)|$.

$$\Re(c \cdot s(y, x)) = \Re(s(y, cx))$$

Using $\otimes$: $\operatorname{Re}\left(s(y, cx)\right)^2 \leq s(cx, cx) s(y, y)$
 $\quad\quad\quad //$
 $\quad\quad\quad |s(y, x)|^2 \quad \underbrace{\bar{c} c}_{=1} s(x, x)$

$$\Rightarrow |s(y, x)|^2 \leq s(x, x) \cdot s(y, y)$$

□