Calculating the kernel of matrix

(an example)

$$A = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 3 & 1 & 4 & 4 \\ 4 & -4 & 0 & 8 \end{pmatrix} \in \mathbb{R}^{3\times 4}$$

$$\ker(A) := \{ x \in \mathbb{R}^4 \mid Ax = 0 \}$$

$$\ker(A) = \ker \begin{pmatrix} 1 & 1 & 2 & 1 \\ 3 & 1 & 4 & 4 \\ 4 & -4 & 0 & 8 \end{pmatrix} \begin{matrix} \\ \leftarrow \text{II} - 3\text{I} \\ \leftarrow \text{III} - 4\text{I} \end{matrix}$$

$$= \ker \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & -2 & -2 & 1 \\ 0 & -8 & -8 & 4 \end{pmatrix} \begin{matrix} \\ \\ \leftarrow \text{III} - 4\text{II} \end{matrix}$$

$$= \ker \begin{pmatrix} \boxed{1} & 1 & 2 & 1 \\ 0 & \boxed{-2} & \boxed{-2} & \boxed{1} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$-2x_2 - 2x_3 + x_4 = 0$

$\Leftrightarrow x_2 = -\alpha + \frac{1}{2}\beta$ ($x_3 = \alpha$, $x_4 = \beta$)

$x_1 + x_2 + 2x_3 + x_4 = 0$

$-\alpha + \frac{1}{2}\beta + \alpha - \beta$ ($x_3 = \alpha$, $x_4 = \beta$)

$x_1 = -\alpha - \frac{1}{2}\beta$

$$= \left\{ \begin{pmatrix} -\alpha - \frac{1}{2}\beta \\ -\alpha + \frac{1}{2}\beta \\ \alpha \\ \beta \end{pmatrix} \;\middle|\; \alpha, \beta \in \mathbb{R} \right\}$$

$$= \left\{ \alpha \cdot \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 1 \end{pmatrix} \;\middle|\; \alpha, \beta \in \mathbb{R} \right\}$$

basis of $\ker(A)$