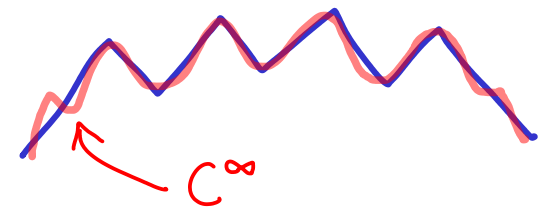


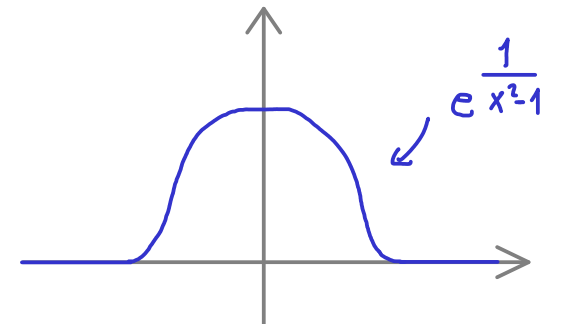
Approximation Theorem



Standard mollifier: C^∞ -function with compact support:

$$\eta: \mathbb{R}^n \longrightarrow \mathbb{R}$$

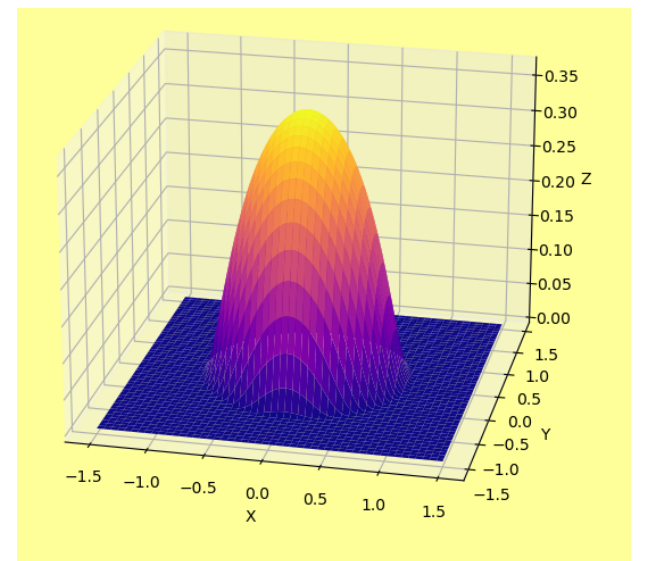
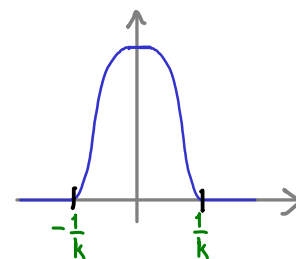
$$\eta(x) := \begin{cases} C \cdot \exp\left(\frac{1}{\|x\|^2-1}\right) & , \|x\| < 1 \\ 0 & , \|x\| \geq 1 \end{cases}$$



where $C > 0$ such that $\int_{\mathbb{R}^n} \eta(x) d^n x = 1$.

Define a Dirac sequence: $\delta_k(x) := k^n \cdot \eta(k \cdot x)$

with three properties:



(1) $\delta_k(x) \geq 0$ for all $k \in \mathbb{N}$, for all $x \in \mathbb{R}^n$

(2) $\int_{\mathbb{R}^n} \delta_k(x) d^n x = 1$ for all $k \in \mathbb{N}$

(3) For all $\epsilon > 0$, $\mathcal{B}_\epsilon(0) := \{y \in \mathbb{R}^n \mid \|y\| < \epsilon\}$ satisfies:

$$\int_{\mathbb{R}^n \setminus \mathcal{B}_\epsilon(0)} \delta_k(x) d^n x \xrightarrow{k \rightarrow \infty} 0$$

Approximation Theorem: Let $f: \mathbb{R}^n \longrightarrow \mathbb{C}$ be continuous.

Then for each compact set $A \subseteq \mathbb{R}^n$: $\|f - \underbrace{\delta_k * f}_{C^\infty}\|_{\infty, A} \xrightarrow{k \rightarrow \infty} 0$

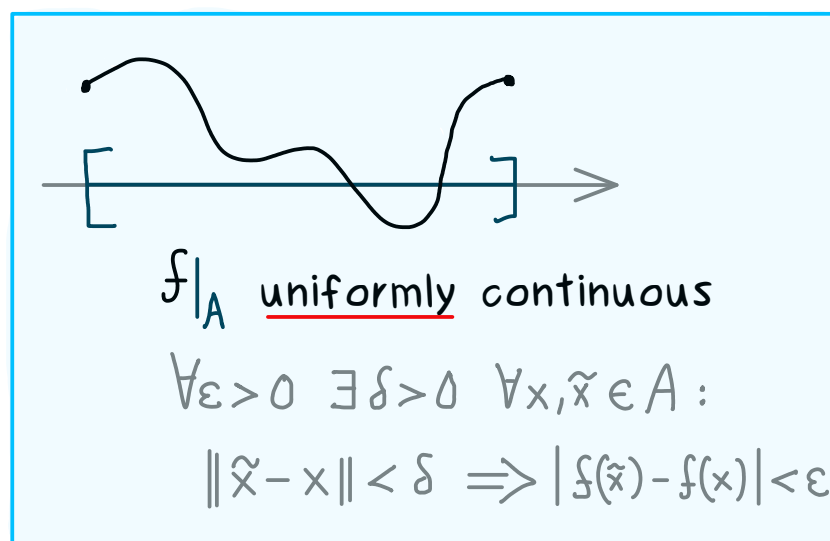
$\left(\delta_k(x) := k^n \cdot \eta(k \cdot x) \right)$

$\| \cdot \|_{\infty, A} = \sup_{x \in A} | \cdot |$

Proof: Let $\varepsilon > 0$ and $A \subseteq \mathbb{R}^n$ be compact.

So we find $\delta > 0$ such that for all $x \in A$
and $y \in \mathcal{B}_\delta(0)$: $|f(x-y) - f(x)| < \varepsilon$

Then for any $x \in A$:



$$|f(x) - (\delta_k * f)(x)| = \left| f(x) - \int_{\mathbb{R}^n} \delta_k(y) f(x-y) d^n y \right|$$

$$= \left| f(x) \underbrace{\int_{\mathbb{R}^n} \delta_k(y) d^n y}_{=1} - \int_{\mathbb{R}^n} \delta_k(y) f(x-y) d^n y \right|$$

$$= \left| \int_{\mathbb{R}^n} \delta_k(y) (f(x) - f(x-y)) d^n y \right| \leq \int_{\mathbb{R}^n} \delta_k(y) |f(x-y) - f(x)| d^n y$$

$$= \int_{\mathcal{B}_{\frac{\varepsilon}{2}}(0)} \delta_k(y) \underbrace{|f(x-y) - f(x)|}_{< \varepsilon} d^n y + \int_{\mathbb{R}^n \setminus \mathcal{B}_{\frac{\varepsilon}{2}}(0)} \delta_k(y) |f(x-y) - f(x)| d^n y$$

$$\leq \varepsilon + \underbrace{\int_{\mathbb{R}^n \setminus \mathcal{B}_{\frac{\varepsilon}{2}}(0)} \delta_k(y) (|f(x-y)| + |f(x)|) d^n y}_{=0 \text{ for all } k \geq K}$$

$$\Rightarrow \sup_{x \in A} |f(x) - (\delta_k * f)(x)| \leq \varepsilon \Rightarrow \|f - \delta_k * f\|_{\infty, A} \xrightarrow{k \rightarrow \infty} 0$$

□

The proof also shows the following for any Dirac sequence:

Approximation Theorem (for bounded functions):

Let $f: \mathbb{R}^n \longrightarrow \mathbb{C}$ be bounded and continuous and δ_k be a Dirac sequence  satisfying (1),(2),(3).

Then for each compact set $A \subseteq \mathbb{R}^n$: $\|f - \delta_k * f\|_{\infty, A} \xrightarrow{k \rightarrow \infty} 0$
 $= \sup_{x \in A} |\cdot|$