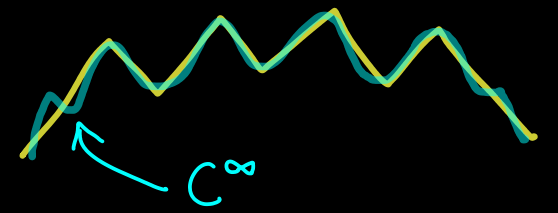


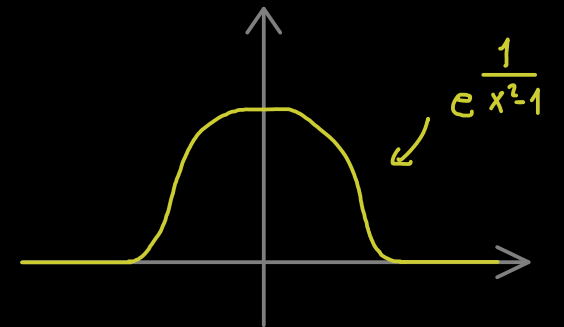
Approximation Theorem



Standard mollifier: C^∞ -function with compact support:

$$\eta: \mathbb{R}^n \longrightarrow \mathbb{R}$$

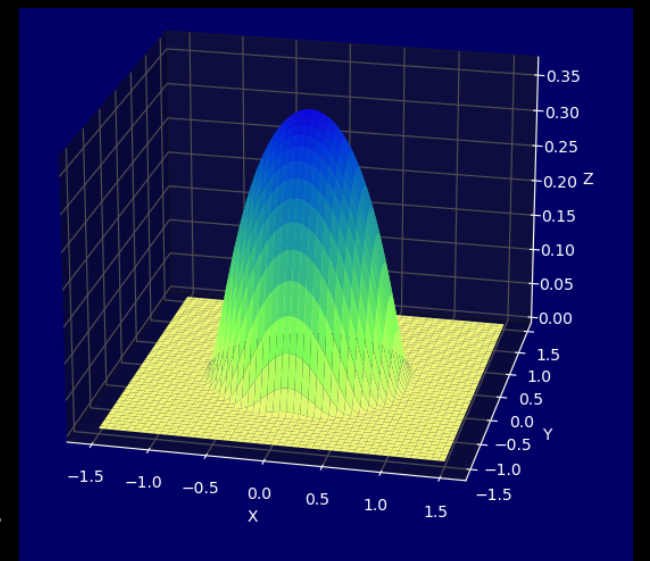
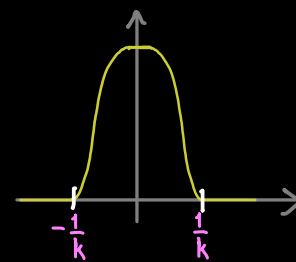
$$\eta(x) := \begin{cases} C \cdot \exp\left(\frac{1}{\|x\|^2-1}\right) & , \|x\| < 1 \\ 0 & , \|x\| \geq 1 \end{cases}$$



where $C > 0$ such that $\int_{\mathbb{R}^n} \eta(x) d^n x = 1$.

Define a Dirac sequence: $\delta_k(x) := k^n \cdot \eta(k \cdot x)$

with three properties:



(1) $\delta_k(x) \geq 0$ for all $k \in \mathbb{N}$, for all $x \in \mathbb{R}^n$

(2) $\int_{\mathbb{R}^n} \delta_k(x) d^n x = 1$ for all $k \in \mathbb{N}$

(3) For all $\epsilon > 0$, $\mathcal{B}_\epsilon(0) := \{y \in \mathbb{R}^n \mid \|y\| < \epsilon\}$ satisfies:

$$\int_{\mathbb{R}^n \setminus \mathcal{B}_\epsilon(0)} \delta_k(x) d^n x \xrightarrow{k \rightarrow \infty} 0$$

Approximation Theorem: Let $f: \mathbb{R}^n \longrightarrow \mathbb{C}$ be continuous.

Then for each compact set $A \subseteq \mathbb{R}^n$: $\|f - \underbrace{\delta_k}_{C^\infty} * f\|_{\infty, A} \xrightarrow{k \rightarrow \infty} 0$

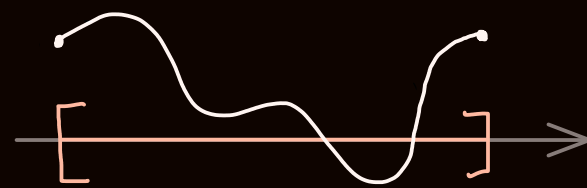
$\left(\delta_k(x) := k^n \cdot \eta(k \cdot x) \right)$

$\| \cdot \|_{\infty, A} = \sup_{x \in A} | \cdot |$

Proof: Let $\epsilon > 0$ and $A \subseteq \mathbb{R}^n$ be compact.

So we find $\delta > 0$ such that for all $x \in A$
and $y \in \mathcal{B}_\delta(0)$: $|f(x-y) - f(x)| < \epsilon$

Then for any $x \in A$:



$f|_A$ uniformly continuous

$\forall \epsilon > 0 \exists \delta > 0 \forall x, \tilde{x} \in A :$

$\|\tilde{x} - x\| < \delta \Rightarrow |f(\tilde{x}) - f(x)| < \epsilon$

$$|f(x) - (\delta_k * f)(x)| = \left| f(x) - \int_{\mathbb{R}^n} \delta_k(\gamma) f(x-\gamma) d^n \gamma \right|$$

$$= \left| f(x) \underbrace{\int_{\mathbb{R}^n} \delta_k(\gamma) d^n \gamma}_{=1} - \int_{\mathbb{R}^n} \delta_k(\gamma) f(x-\gamma) d^n \gamma \right|$$

$$= \left| \int_{\mathbb{R}^n} \delta_k(\gamma) (f(x) - f(x-\gamma)) d^n \gamma \right| \leq \int_{\mathbb{R}^n} \delta_k(\gamma) |f(x-\gamma) - f(x)| d^n \gamma \quad (1)$$

$$= \int_{\mathcal{B}_{\frac{\delta}{2}}(0)} \delta_k(\gamma) \underbrace{|f(x-\gamma) - f(x)|}_{< \epsilon} d^n \gamma + \int_{\mathbb{R}^n \setminus \mathcal{B}_{\frac{\delta}{2}}(0)} \delta_k(\gamma) |f(x-\gamma) - f(x)| d^n \gamma$$

$$\leq \epsilon + \int_{\mathbb{R}^n \setminus \mathcal{B}_{\frac{\delta}{2}}(0)} \delta_k(\gamma) (|f(x-\gamma)| + |f(x)|) d^n \gamma$$

$= 0$ for all $k \geq K$

$$\Rightarrow \sup_{x \in A} |f(x) - (\delta_k * f)(x)| \leq \epsilon \Rightarrow \|f - \delta_k * f\|_{\infty, A} \xrightarrow{k \rightarrow \infty} 0$$

□

The proof also shows the following for any Dirac sequence:

Approximation Theorem (for bounded functions):

Let $f: \mathbb{R}^n \longrightarrow \mathbb{C}$ be bounded and continuous and δ_k be a Dirac sequence ^{satisfying (1),(2),(3)}.

Then for each compact set $A \subseteq \mathbb{R}^n$: $\|f - \delta_k * f\|_{\infty, A} \xrightarrow{k \rightarrow \infty} 0$
 $\underbrace{\| \cdot \|_{\infty, A}}_{= \sup_{x \in A} | \cdot |}$