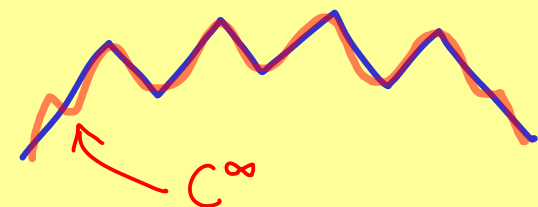


The Bright Side of Mathematics



Approximation Theorem

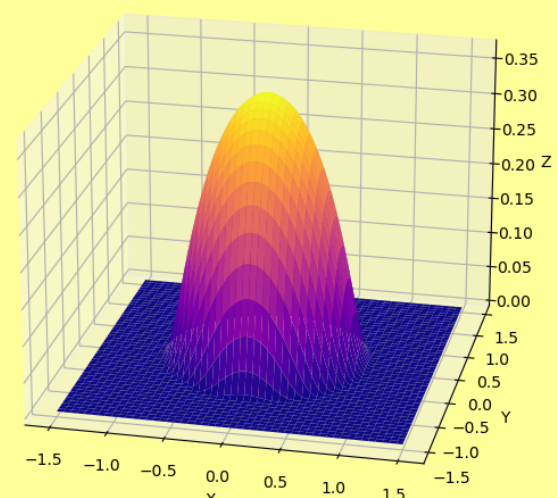
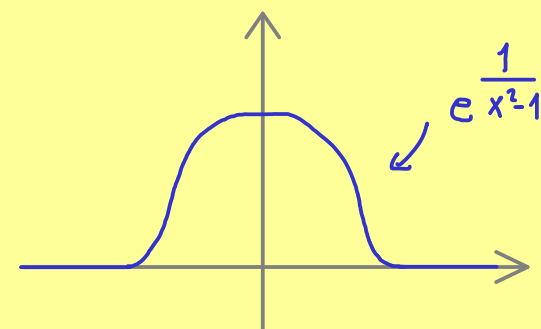


Standard mollifier: C^∞ -function with compact support:

$$\eta: \mathbb{R}^n \rightarrow \mathbb{R}$$

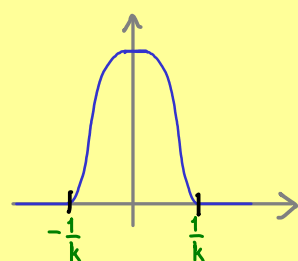
$$\eta(x) := \begin{cases} C \cdot \exp\left(\frac{1}{\|x\|^2-1}\right) & , \|x\| < 1 \\ 0 & , \|x\| \geq 1 \end{cases}$$

where $C > 0$ such that $\int_{\mathbb{R}^n} \eta(x) d^n x = 1$.



Define a Dirac sequence: $\delta_k(x) := k^n \cdot \eta(k \cdot x)$

with three properties:



(1) $\delta_k(x) \geq 0$ for all $k \in \mathbb{N}$, for all $x \in \mathbb{R}^n$

(2) $\int_{\mathbb{R}^n} \delta_k(x) d^n x = 1$ for all $k \in \mathbb{N}$

(3) For all $\epsilon > 0$, $\mathcal{B}_\epsilon(0) := \{y \in \mathbb{R}^n \mid \|y\| < \epsilon\}$ satisfies:

$$\int_{\mathbb{R}^n \setminus \mathcal{B}_\epsilon(0)} \delta_k(x) d^n x \xrightarrow{k \rightarrow \infty} 0$$

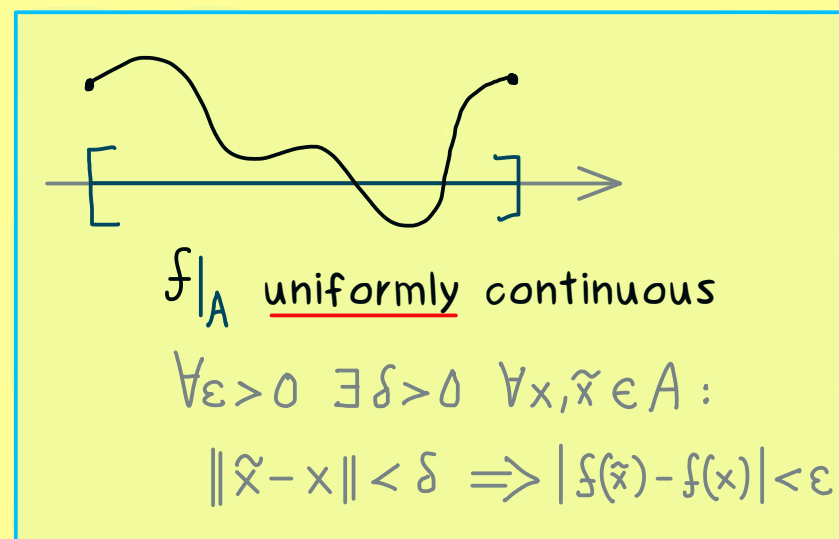
Approximation Theorem: Let $f: \mathbb{R}^n \rightarrow \mathbb{C}$ be continuous.

Then for each compact set $A \subseteq \mathbb{R}^n$: $\|f - \underbrace{\delta_k * f}_{C^\infty}\|_{\infty, A} \xrightarrow{k \rightarrow \infty} 0$
 $\left(\delta_k(x) := k^n \cdot \eta(k \cdot x) \right)$
 $\underbrace{\quad}_{C^\infty} = \sup_{x \in A} |\cdot|$

Proof: Let $\epsilon > 0$ and $A \subseteq \mathbb{R}^n$ be compact.

So we find $\delta > 0$ such that for all $x \in A$
 and $y \in \mathcal{B}_\delta(0)$: $|f(x-y) - f(x)| < \epsilon$

Then for any $x \in A$:



$$\begin{aligned} |f(x) - (\delta_k * f)(x)| &= \left| f(x) - \int_{\mathbb{R}^n} \delta_k(y) f(x-y) d^n y \right| \\ &= \left| f(x) \underbrace{\int_{\mathbb{R}^n} \delta_k(y) d^n y}_{=1} - \int_{\mathbb{R}^n} \delta_k(y) f(x-y) d^n y \right| \\ &= \left| \int_{\mathbb{R}^n} \delta_k(y) (f(x) - f(x-y)) d^n y \right| \leq \int_{\mathbb{R}^n} \delta_k(y) |f(x-y) - f(x)| d^n y \\ &= \int_{\mathcal{B}_{\frac{\epsilon}{2}}(0)} \delta_k(y) \underbrace{|f(x-y) - f(x)|}_{< \epsilon} d^n y + \int_{\mathbb{R}^n \setminus \mathcal{B}_{\frac{\epsilon}{2}}(0)} \delta_k(y) |f(x-y) - f(x)| d^n y \\ &\leq \epsilon + \underbrace{\int_{\mathbb{R}^n \setminus \mathcal{B}_{\frac{\epsilon}{2}}(0)} \delta_k(y) (|f(x-y)| + |f(x)|) d^n y}_{=0 \text{ for all } k \geq K} \end{aligned}$$

$$\Rightarrow \sup_{x \in A} |f(x) - (\delta_k * f)(x)| \leq \epsilon \Rightarrow \|f - \delta_k * f\|_{\infty, A} \xrightarrow{k \rightarrow \infty} 0$$

□

The proof also shows the following for any Dirac sequence:

Approximation Theorem (for bounded functions):

Let $f: \mathbb{R}^n \rightarrow \mathbb{C}$ be bounded and continuous and δ_k be a Dirac sequence.

Then for each compact set $A \subseteq \mathbb{R}^n$: $\|f - \delta_k * f\|_{\infty, A} \xrightarrow{k \rightarrow \infty} 0$
 $\underbrace{\quad}_{= \sup_{x \in A} |\cdot|}$

satisfying (1),(2),(3)